# LENS DESIGN USING GROUP INDICES OF REFRACTION

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# ABSTRACT

An approach to lens design is described in which the ratio of the group velocity to the speed of light (the group index) in glass is used, in conjunction with the more familiar phase index of refraction, to control certain chromatic properties of a system of thin lenses in contact. It is shown that at the wavelength of a maximum or minimum (where the phase power of a lens is locally independent of wavelength), the group power is equal to the phase power. It is further shown that in a lens consisting of three or more elements, the phase and group powers can be constrained to be both equal and independent of wavelength (achromatic) at one or more wavelengths. In the neighborhood of such wavelengths, both the first and second derivatives of phase power with respect to wavelength are zero, giving this type of lens (in principle) an exceptionally high degree of achromatism not previously described, herein called group-achromatism. The first-order design of thin-lens systems is illustrated by examples with the help of a computer program incorporating the methods described.

Key words: lens design, group velocity, achromatism, secondary color, refraction, dispersion, optical glass, interferometry, ultrashort pulses

## INTRODUCTION

The index of refraction normally tabulated in glass catalogs and used in lens design is, of course, the phase index that ordinarily appears in Snell's law of refraction. In a glass whose (phase) index at a particular wavelength is $\mu$, the phase of a light signal propagates at the speed $c/\mu$, where c is the speed of light in vacuum. However, in a dispersive medium, a wave-packet or photon carries energy (or information) at a speed $c/\gamma$, the group velocity[1-4], where the group index of refraction (the ratio of the group velocity to c) is related to the phase index through the differential equation

$$\gamma = \mu - \lambda\,(d\mu/d\lambda). \qquad (1)$$

Two derivations of Eq. 1 are given in the Appendix. Since $d\mu/d\lambda$ is always negative in glass, the group index for any glass is always larger than its phase index (except in the vicinity of a resonance).

The difference between the two indices $\mu$ and $\gamma$ is a measure of the dispersive power of the glass. In lens design[5] the dispersive power of glass (relative to its refractive power) is customarily represented by the Abbe number or constringence $\nu$, and the partial dispersion P, computed for example (for a particular wavelength range near the Fraunhofer d line) from the formulas

$$\nu_d = (\mu_d - 1)/(\mu_F - \mu_C), \qquad (3a)$$

$$P_d = (\mu_d - \mu_F)/(\mu_F - \mu_C). \qquad (3b)$$

However, it is possible, computationally convenient, and instructive to make explicit use of the group index of refraction of glass together with the more conventional phase index in designing achromatic lenses. (In this article, the term achromat is meant to include all types of multi-element lenses in which the chromatic variation of power is controlled. ) As will be shown, it is possible to control not only the power of a lens system at a selected set of discrete wavelengths (as in standard methods). but also (using both phase and group indices) to control directly the location of local maximums or minimums (in other words turning points), where the power of the lens is locally independent of wavelength, as well as to control the location of stationary points. where both the first and the second derivative of the phase power of a compound lens is zero.

## TWO-ELEMENT LENS DESIGN

To design a two-element achromat it is customary to constrain the power of the lens combination at two separate wavelengths, requiring the lens to have the same power at both wavelengths. Here, we investigate the design of a two-element thin-lens achromat in which the phase power $\Pi$ and the group power $\Gamma$ are required to be equal to each other at one and the same wavelength.

The phase power is the curvature of a wavefront when it emerges from a lens illuminated by a distant point source. To visualize the physical meaning of group power, imagine the lens to be illuminated by an extremely short pulse of light from the same distant source. The group power is the curvature of the surface of maximum energy density in space, as the pulse leaves the lens. The group power of a single lens element is always greater than the phase power, In a lens consisting of more than one element. the group power can be greater than, smaller than. or equal to the phase power at different wavelengths, depending upon the construction of the lens.

Since the power of two thin lenses in contact is the sum of the powers of the individual elements, we have for such a two-element lens with equal phase and group power

$$\Pi = (\mu_1 - 1)\phi_1 + (\mu_2 - 1)\phi_2 = (\gamma_1 - 1)\phi_1 + (\gamma_2 - 1)\phi_2, \tag{4}$$

where the subscripts refer to the first and second elements and $\phi$ is the difference between the curvatures of the two surfaces of each thin-lens element.

If we define, analogously to the Abbe number, a number

$$G \equiv \frac{\gamma - 1}{\gamma - \mu} \tag{5}$$

we obtain from Eq. 4, after some algebra, the result

$$\phi_1(\gamma_1 - 1) = -\Pi\left(\frac{G_1}{G_2 - G_1}\right) \tag{6a}$$

and

$$\phi_2(\gamma_2 - 1) = \Pi\left(\frac{G_2}{G_2 - G_1}\right) \tag{6b}$$

These expressions are formally similar to the classical definition of a two-element achromat, except that group indices and G numbers are used in place of phase indices and Abbe numbers. Like the Abbe number, G measures the refractive power relative to the dispersive power of the glass. In the conventionally designed lens the power of the achromat will be the same at the two wavelengths for which the Abbe numbers were calculated (for example, the d and F lines), with a turning point somewhere between them. in the lens defined by Eq. 4, a turning point is located at the wavelength for which the phase and group indices were calculated, where its power will have the specified value. The reason for ibis coincidence will be clear when the general case of an N-element lens is considered.

The G number defined by Eq. 5 is proportional to the spectral V number $V_\lambda$ defined by J I Rayces[6], the relation between these parameters being

$$G = 1 + \left(\frac{2}{\lambda}\right) V_\lambda$$

$V_\lambda$ has the dimension of length, whereas G is dimensionless

## N-ELEMENT LENS DESIGN

For a lens consisting of N thin elements in contact, the total phase power of the combination, at the jth wavelength, is

$$H_j = \sum_{i=1}^{N} (\mu_{ij} - 1)\phi_i \,, \tag{7}$$

while the group power at the same wavelength is

$$\Gamma_j = \sum_{i=1}^{N} (\gamma_{ij} - 1)\phi_{ij} \tag{8}$$

If the phase and group powers are constrained to be equal at the jth wavelength then

$$0 = \sum_{i=1}^{N} (\gamma_{ij} - \mu_{ij})\phi_i \tag{9}$$

(Note that if a turning point is to be located at the jth wavelength then by definition, and using Eq 7, we have the condition

$$\partial \Pi_j / \partial\lambda = \sum (\partial\mu_{ij} / \partial\lambda)\phi_{ij} = 0$$

nut by virtue of Eq 1. this condition leads directly to Eq. 9. In other words, a turning point in phase power inevitably occurs at the jth wavelength when the phase power is equal to the group power there. )

$$S_j = \frac{\partial^2 \Pi_j}{\partial\lambda_j^2} = \sum \frac{\partial^2 \mu_{ij}}{\partial\lambda^2}\phi_i = \left(\frac{1}{\lambda_j^2}\right)\sum\left[\left(\gamma_{ij} - \mu_{ij}\right) - \lambda_j\left(\frac{\partial(\gamma_{ij} - \mu_{ij})}{\partial\lambda_j}\right)\right]\phi_i , \qquad (10)$$

where the summations are taken over the N elements. Denoting the dimensionless quantity in square brackets by $A_{ii}$, we have, for the general form of secondary color.

$$S_j = \frac{1}{\lambda_j^2}\sum \Delta_{ij}\phi_i \qquad (11)$$

This will be a minimum or a maximum if, in addition to the previously described constraints (following the standard least square procedure), we require, for any k (k 1.2. 3... N).

$$\frac{\partial}{\partial\phi_k}\left(S_j^2\right) = 0 \quad or \quad \frac{\partial}{\partial \Delta_{kj}}\left(S_j^2\right) = 0 \qquad (12)$$

For the simplest case of N 2, one can extend Eq 11

$$S_j^2 = (\Delta_{1j}^2 \phi_1^2 + 2\Delta_{1j}\Delta_{2j}\phi_1\phi_2 + \Delta_{2j}^2 \phi_2^2) / \lambda_j^2 \tag{13}$$

In this case it is easily shown that either version of Eq 12 leads to the mull

$$\Delta_{1j}\phi_1 = \Delta_{2j}\phi_2$$

Making use of Eqs. 610 constrain the solution for a binary achromat to place a local extremum at $\lambda_j$, we arrive at the result (for this case)

$$\frac{A_{1i}}{\gamma_{1i} - \mu_{1i}} = \frac{A_{21}}{\gamma_{2i} - \mu_{2i}} \tag{14}$$

In general let us define

$$\Theta_{ij} = \frac{\Delta_{ij}}{\gamma_{ij} - \mu_{ij}} \tag{15}$$

for the ith element, as a measure of the rate of change of dispersive power with wavelength relative to the dispersive power itself. The parameter Θ is proportional to the spectral relative partial dispersion $P_\lambda$ defined by Rayces[5], the relation being

$$\Theta = 4\lambda P_\lambda$$

Θ is dimensionless whereas $P_\lambda$ has the dimension of inverse length

From Eq. 14 it follows that *secondary color at an extremum will be a minimum* if the two glasses in a binary achromat have the same value of $\Theta$. As is well known, at a given $\Theta$ (or P) the available range of G (or V) is quite small.

Returning to the general case of N elements, it follows from Eqs. 12 and 11 that if secondary color (as the term is here used) at the jth wavelength is a minimum, then at that wavelength, for any k,

$$S_j \Delta_{kj} = 0. \qquad (16)$$

Since $\Delta_{kj}$ is never zero, it follows that Eq. 16 is equivalent to the requirement

$$\sum \Delta_{ij} \phi_i = 0. \qquad (17)$$

This means that for a given set of glass types in a lens, we can choose lens curvatures that minimize secondary color at some wavelength (satisfying Eq. 17). which may also be the location of a turning point (satisfying Eq. 9). For N>2, this leaves N- 2 additional constraints to be imposed to fully define the lens.

Eq. 17 states that if the second derivative, $S_j$, of the dioptric power of the lens is a minimum at the jth wavelength, *the second derivative itself is zero* at that wavelength. This might have been expected, given the fact that, since $S_j$ can be either positive or negative, the only way its absolute value (or $S_j^2$) can be a minimum is for $S_j$ itself' to be zero.

If the phase power is equal to the group power (so that the phase power has zero slope) at $\lambda_j$, and if the second derivative S, is also zero there, then the jth wavelength will be neither a maximum nor a minimum point. but the location of a horizontal inflectional tangent, which could be called a stationary point, at which the secondary color is zero.

If it should happen that $\Theta_{ij}$ is the same for all of the elements (not easily achieved with real glass), then from Eq. 15 it follows that Eq. 17 would be merely a constant multiple of Eq. 9.

In that case, if Eq. 17 (minimizing secondary color) is used as a constraint at the jth wavelength, a stationary point will automatically be located there, so that Eq 9 could not be used as a separate constraint at the same wavelength in this case (however, it could be used at a different wavelength).

In the two-element lens considered previously, we minimized secondary color (in principle) *through the independent choice of glass types to* satisfy Eq. 14, leaving the choice of curvatures of the elements free to satisfy the two possible design constraints, namely Eqs. 7 and 9 at one or two wavelengths,

in the design of a real lens using real glass it is not generally possible to find several glass types having almost the same $\Theta_{ij}$ and simultaneously significantly differing G numbers (the latter required to keep the powers of the lens elements from becoming excessively large). Nevertheless, we see that in a multi-element lens (N>2), it is possible to eliminate secondary color in the neighborhood of a turning point through the independent constraint represented by Eq. 17. In this case, it is desirable to choose glass types in which the $\Theta_{ij}$ values are as diverse as possible, maximizing det[A] (see discussion of matrix solution following).

In most applications, although it is possible, it would hardly be useful to minimize secondary color (as the term is used here, meaning the second derivative of lens power with respect to wavelength) at some wavelength, unless that wavelength were also the location of an extremum.

## SOLUTION OF THE MATRIX EQUATION

For a system of N elements, we can impose a total of N constraints, which may be of the form given by Eqs. 7, 8, 9 or 17 or a combination of these At least one of the constraints must specify the power of the system at some wavelength. The system of linear equations can be expressed as a matrix equation

$$A\phi = p \tag{18}$$

where p is an N-dimensional vector whose components are either zero (for a constraint expressed by Eq. 9 or Eq. 17) or the targeted values of the lens power under each other constraint. $\phi$ is the vector of lens curvatures, The matrix elements $[A_{ij}]$ are the coefficients of $\phi_i$ in Eqs. 7, 8, 9 or 17 depending upon the designer's choice of the jth constraint The solution is obtained by computing the inverse of the N x N matrix A. Thus

$$\phi = A^{-1} p \tag{19}$$

A solution is possible only if the glass matrix A is not singular, i.e. only if

$$\det[A] \neq 0. \tag{20}$$

Otherwise the powers of the lens elements would be infinite.

To minimize the powers of the lens elements it is desirable to choose glasses such that $|\det[A]|$ is as large as possible, or at least not too small. since larger $\det[A]$ will keep smaller the powers of the lens elements. The glass matrix is a (unction only of the parameters of the glass, bill it dots depend upon whether the constraints imposed arc of the kind given by Eq. 7, 8, 9 or 17 (or some other kind not explicitly considered here), and their wavelengths. A brute force approach to the initial selection of glass types for an N-element lens with any given set of constraints, feasible with a computer if the number of candidate glasses is not too large, would be to actually calculate the value of $|\det[A]|$ for all $M!/((M - N)!N!)$ distinct combinations of M different candidate glass types. taken N at a time, and to favor the combinations giving larger values. For M = 51 and N = 3, the calculation of $|\det[A]|$ for all 20,825 combinations to find the best 20 takes only two minutes on my 25 Mhz laptop computer.

A more intuitive approach follows if det[A] is given a geometrical interpretation. Thus, for a 2-element achromat constrained by Eq. 7 and Eq 9. let

$$u_1 = (\mu_1-1)\mathbf{i} + (\gamma_1-\mu_1)\mathbf{j} \quad \text{and} \quad u_2 = (\mu_2-1)\mathbf{i} + (\gamma_2-\mu_2)\mathbf{j}$$

be the pair of two-dimensional vectors extending from the origin to the two glass points in the plane defined by orthogonal axes ($\mu$-1), ($\gamma$-$\mu$) in Figure 1. It will be seen that the length of the vector cross-product $u_1 \times u_2$ is just the area of the parallelogram (or twice the area of the triangle) defined by the two vectors. It is easily shown that $|u_1 \times u_2|$ is identical to $|\det[A]|$. The selection of glass types for the achromat will be optimized by maximizing this area, or det[A]. A similar interpretation applies if, instead of Eq. 9, some other second constraint is used. The corresponding functions of glass indices (evaluated at one or two wavelengths) would then become the coordinate axes in a diagram similar to Figure 1.

In the case of three elements, a simple geometrical interpretation is likewise possible. In this case three vectors $u_1$, $u_2$ and $u_3$ from the origin represent the coordinates of the three glass types in a 3-space whose axes are the coefficients of $\phi$ in any three of the constraints (Eqs. 7.8, 9, or 17), one of which must be either of the first two. It follows that the triple product

$$|u_1 \times u_2 \cdot u_3|,$$

or any permutation thereof, is the volume of the three-dimensional parallelepiped defined by the three vectors, and identical to $|\det[A]|$. The selection of glass types for a lens triplet having given constraints will be optimized by making this volume as large as is practical.

In the general case of N elements, it can be conjectured that $|\det[A]|$ represents a volume (which the lens designer might seek to maximize) in an N-dimensional orthogonal vector space whose axes depend upon the types and wavelengths of the constraints imposed upon the lens.

Since this is not easy or even possible to visualize, actual calculation of det[A] for various glass combinations under consideration may offer the best guidance in this case.

The problem of designing a thin lens consisting of N elements to satisfy N + 1 constraints through the judicious choice of N glasses (such a lens being called by Herzberger a superachromat in the case N = 3) has been studied extensively. Following, for example, Herzberger[7], Herzberger & McClure[8], Lessing[9], Hoogland[10], Sigler[11] and Maxwell[12], one WCS that formally the problem can be addressed by considering the design of au (N+1)-element lens, in which by selection of N glasses the power of one of the elements is brought to zero, or nearly so, in which case that element can be omitted. Using Eq. 19 this would be amenable to a brute force solution akin to the evaluation of det[A] previously discussed, and would be applicable under any of the constraints described. Whether such an approach would lead to interesting results is an open question that is beyond the scope of this investigation.

## EXAMPLES

A FORTRAN program for designing multi-element thin-lens achromats was created in order to implement and test the general approach and specific types of constraints considered in this paper. Sellmeier coefficients for the selected glasses arc obtained from a computer-readable form of the Schott Optical Glass Catalog (supplemented with special glasses) and used to compute phase indices at the required wavelengths Group indices arc computed by numerical different iation of the phase index with respect 10 wavelength and by using Eq. 1. The matrix equation (Eq. 18) is solved by standard procedures[13]. The phase and group powers of the lens combination arc then computed over a range of wavelengths for graphical display.

Three types of constraints arc used, from among the four described in the preceding section. Those used arc summarized in Table 1. They can be imposed in any combination and in any order up to a total of N constrains to define a lens consisting of N elements, provided the dioptric power of the lens is defined for at least one wavelength by a type 1 constraint.

Examples of thin-lens achromats designed using the methods described are illustrated in Figs. 2-5. The glasses selected for these examples were chosen without consideration of their spectral transmittance and other physical properties that might affect their suitability for building an actual lens. Likewise, the control of spherical aberration and spherochromatism is ignored here.

The first example (Figure 2 and Table 2) is a two-element achromat constrained by Eq. 7 to give f = 100 at 600" nm and constrained by Eq. 9 to have a turning point at 500 nm. At the extremum, the focal length (f = 99.9275, not independently constrained) computed from the group index is equal to the focal length computed from the phase index, in compliance with the type 2 constraint applied at 500 nm. Note that where the phase power of the lens is increasing with wavelength, the group power is weaker than the phase power, and *vice versa*. In this and other examples, at wavelengths far from the stationary point(s), the group power departs substantially from the phase power of the lens.

The second example (Figure 3 and Table 3) is a three-element lens constrained by Eqs. 9 and 17 at 500 nm, where the lens is also constrained by Eq. 7. The three glasses are those used for illustration of an apochromat by Kingslake[5]. Here the example illustrates the flexibility afforded by the method under discussion to target the wavelengths of extrema and stationary points and, if desired, to control the power of the system at those points.

When the lens in Table 3 is corrected for spherical aberration at 500 nm as an F/7.8 cemented triplet, the design achieves a Strehl ratio of 0.9 or greater for wavelengths from about 460 nm to 570 nm. Over that wavelength range the chromatic variation of relative phase-focal length is about $10^{-5}$. This is less than the diffraction depth of focus (and therefore of no practical significance) unless the lens is scaled to about 780 mm (30 inches) or larger aperture at F/7.8. The wavelength range over which the lens is effectively group-achromatic is considerably smaller, so that the design could (for an application requiring this property) be appropriate for a smaller lens.

Figure 4 compares the foregoing lcns with a triplct madc of the same materials but constraincd only by [q. 7 at three closely spaced wavclengths (480, 500 and 520 rim). The results arc practically the same, but remain fundamental ly different. If the three wavelengths arc brought closer and closer togcther, the glass matrix [ A ] will become progressively more poorly conditioned unt i 1, in the limit, the solution becomes i ndeterminate

The third example (Figure 5 and '1'able 4) i llustrates the power of the matrix approach used here to prc-design chromatically controlled thin lcns systems consisting of many elements, in this case eight. In the example, type 2 constraints arc placed at 500, 600, 700, and 800 nm, and the focal length of the lens is 1000 at each of those wavelengths. Clearly, at least for paraxial rays or in the absence of spherochromatism and other aberrations, such a lens would perform very well from around 460 to nearly 1000 nm, if it could bc built to satisfactory tolerances.

## DISCUSSION

The approach to first-order thin lens design considered in this paper might bc called the $\gamma$-$\mu$ method, since the difference between the group and phase i ndices is used explicitly in controlling the wavelength variation of the lens power. In making usc of infinitesimal derivatives of the phase index $\mu$ with respect to wavelength instead of glass properties defined as ratios of finite differences, the method is closely related to that of Rayces[6]. However, the explicit usc of group indices of refraction in lens design has not been described or suggested previously as far as I know. It offers some particularly interesting features (1) Like Rayces's approach using $V_\lambda$ and $P_\lambda$, the $\gamma$-$\mu$ method provides a definite, calculable and physically meaningful measure of the dispersion of an optical glass at any wavelength, avoiding the arbitrariness inherent in classical glass parameters such as Abbe number and partial dispersion: (2) the approach therefore affords a notational and computational simplicity well suited to implementation in computer programs (as exemplified in the matrix method demonstrated here, in which. as shown, the determinant of the glass matrix. det [ A ], has a rigorous geometrical interpretation;); (3) by using the $\gamma$-$\mu$

method, a designer can constrain specifically the wavel engths at which local extrema and stationary points occur (where, as shown, the phase and group dioptric powers of a multi-element lens are identical). instead of targeting the power at neighboring wavelengths and adjusting the stationary points (if necessary) by trial and error; and (4) using the $y$-$\mu$ method we have shown that secondary color at an extremum can be minimized (made zero) in a multi-element lens (N> 2) by imposing a definite independent constraint involving the curvatures of the individual elements, of the form given by Eq. 17.

It is interesting to note that. in the neighborhood of such a point (as in Example 2), the group power of the lens also has a local extremum, so that both are locally independent of wavelength (i .e., *the lens is achromatic and con focal in both phase and group power*). Such a lens seems not to have been described previously. This special property might be termed group-achromatism, for lack of a better name. In the neighborhood of the \\ avelength for which a lens is group-achromatic, and only in this case, a photon (or wave packet of finite length and non-zero bandwidth) from a distant point source will arrive at the focal point not only with the same phase (as required to form a diffraction-limited image), but *at one and the same time*, no matter where within the aperture it might have pissed through the lens (in the absence of spherical and other aberrations). This characteristic is normally of no practical significance for an imaging system, but it might be a consideration in certain applications involving broad-band interferometry where fringe visibility is an issue, or in systems for focusing ultrashort light pulses, both being cases where an optical system might need to be exquisitely achromatic.

While the thin lens approximation can be useful in starling the design of a thick lens. ultimately the lens must be optimized by exact raytracing. The group index of refraction, and the concept of group power (as well as group aberrations) can be applied in exact raytracing which (using in Snell's law the group index instead of the phase index) will give the paths normal to a surface of maximum energy density propagating through a system. Thus, it would be possible to control group-aberrations selectively in a thick lens system, using for example Glatzel's adaptive method of optimization as developed by Rayces[14] and applied in EIKONAL.

Apart from any practical value the $y$-$\mu$ method might offer, it elucidates, perhaps for the first time, the physical significance of group velocity in the performance of a lens system

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APPENDIX

# GROUP VELOCITY IN GEOMETRICAL OPTICS

While the concept of group velocity is familiar in wave mechanics and electrical engineering (e.g. the theory of transmission lines and wave guides), the notion is not ordinarily encountered in the context of geometrical optics. The following derivations of Eq. 1 are offered for the convenience of the reader,

## *1. Wave derivation*

Consider an amplitude-modulated tight signal propagating within a dispersive medium, Assume the carrier to have a constant monochromatic frequency $\omega = 2\pi\nu = 2\pi c/\lambda$ radians/see, while the modulation takes place with a single frequency $\Delta\omega << \omega$. If the carrier is a linearly polarized plane wave traveling in the + x direction at speed $c/\mu$, the transverse electric field can be represented as a function of time t and position x by the complex amplitude

$$E^{carrier} = Ae^{i\omega(t - x\mu/c)} \tag{A11}$$

In Eq. A 1.1, c is the speed of light in vacuum, and $\mu$ is the refractive index of the medium for light of wavelength $\lambda$. As shown in textbooks on Fourier analysis, the sinusoidal amplitude modulation gives rise to sidebands at the frequencies $\omega+\Delta\omega$ and m Am above and below the carrier frequency, the corresponding electric fields having the complex amplitudes

$$E^{+} = Be^{i(\omega+\Delta\omega)[t-(\mu+\Delta\mu)x/c]} \text{ and } E^{-} = Be^{i(\omega-\Delta\omega)[t-(\mu-\Delta\mu)x/c]} \tag{A1.2}$$

B is the amplitude of the modulation, and the variation of $\mu$ with $\omega$ is taken into account. The total electric field at (x, t) is thus

$$E^{total} = E^{carrier} + E^{+} + E^{-}. \tag{A1.3}$$

Expanding the products of the quantities in parentheses in Eq A 1.2, neglecting terms of order $\Delta\omega\Delta\mu$, and factoring out the carrier signal, one can write Eq. A 1.3 as

$$E^{total} = e^{i\omega(t - \mu x/c)}\left[A + 2B\cos(\Delta\omega\, t - (\omega\,\Delta\mu + \mu\,\Delta\omega)\, x/c)\right]. \tag{A 1.4}$$

The cosine-term represents a signal of amplitude B traveling in the +x direction at speed x/t = $v_g$, the group velocity, where (in the limit as $\Delta\omega \to 0$)

$$\frac{c}{v_g} = \mu + \omega \frac{\partial\mu}{\partial\omega} \equiv \gamma. \tag{A1.5}$$

But since $\partial\omega/\partial\lambda = (2\pi c/\lambda^2)$, Eq. A 1,5 is identical to Eq. 1 The derivation brings out the fact that whereas a surface of constant phase (a wavefront) propagates at speed c/p, the information carried by an amplitude-modulated light beam propagates at a speed dependent upon the wavelength-variation of the refractive index of the medium, namely the group velocity c/y.

## 2. *Derivation from the viewpoint of interferometry*

In a Twyman-Green interferometer (or any two-beam interferometer) with equal (i.e. balanced) thicknesses of dispersive material in the two arms, white-light fringes (fringes of zero order) can be observed only if the path lengths in the two arms are made equal to within a tolerance that is approximately equal to the coherence length of the light ( ~ $\lambda^2/\Delta\lambda$).

The fringes of zero order will disappear when a dispersive plate is introduced into only one of the arms. However, if the bandwidth is sufficiently narrow or the plate sufficiently thin. the fringes can be restored by suitably shortening the path length in air in the same arm (or lengthening the path in the other). The amount, Al.. by which it is necessary to shorten (or lengthen) the path in air can be calculated by the following physical considerations[15].

First, suppose that the plate is made of an imaginary material whose refractive index $\mu$ is the same at all wavelengths. In this case, the white- light fringes will be restored if the air path length in the arm containing the material is shortened by exactly an amount $(\mu - 1)\ell$, w'here f' is the thickness of the plate, Let it be supposed that this adjustment has been made and the fringes are again visible in light of finite bandwidth centered at wavelength $\lambda_0$.

Next suppose that, without changing the value of the refractive index at $\lambda_0$, the dispersion of the material is restored so that $\partial\mu/\partial\lambda$ is no longer zero, Now, for any two wavelengths $\lambda_1$ and $\lambda_2$ near $\lambda_0$ separated by a wavelength difference $\sigma = \lambda_2 - \lambda_1$, there will arise a phase difference given by

$$\Delta\psi = \left(\frac{2\pi \ell\sigma}{\lambda_0}\right)\frac{\partial\mu}{\partial\lambda} \tag{A2.1}$$

Because of this phase difference (or phase gradient as a function of wavelength), the light amplitudes summed over differing wavelengths will no longer add constructively to form fringes of maximum visibility.

But now suppose that an additional air path $\delta$ be introduced into the same arm. Between the two wavelengths previously considered, this additional air path will contribute a new phase difference given by

$$\Delta\psi' = 2\pi\delta\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) = \left(\frac{2\pi\delta\sigma}{\lambda^2}\right) \tag{A2.2}$$

It will be apparent that to restore constructive interference the additional air path $\delta$ should be chosen to be just that required to compensate for the dispersion of the glass plate; that is.

$$\Delta\psi' = -\Delta\psi . \tag{A2.3}$$

It follows that

$$\delta = -\lambda_0 \ell \frac{\partial\mu}{\partial\lambda} . \tag{A2.4}$$

The total air path correction to compensate for the introduction of the glass plate is thus

$$\Delta L = (\mu - 1)\ell + \delta = (\gamma - 1)\ell , \tag{A2.5}$$

where $\gamma$ is duly given by Eq. 1. From the wave derivation it was evident that $c/\gamma$ is the velocity of propagation of a wave packet. The present derivation shows, then, that fringe visibility is restored when the paths are adjusted so that a wave packet or photon traversing the two-beam interferometer arrives at the detector simultaneously by both paths. The experiment considered would provide a direct measurement of the index $\gamma$ at $\lambda_0$.

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TABLE I

LENS DESIGN CONSTRAINTS

| constraint type | description | definition |
|---|---|---|
| 1 | dioptric power fixed at $\lambda$ | equation 7 |
| 2 | first derivative zero at $\lambda$ | equation 9 |
| 3 | second derivative zero at $\lambda$ | equation 17 |

TABLE 2

OPTICAL CONSTANTS AND CONSTRUCTION PARAMETERS FOR A

TWO-ELEMENT THIN-LENS ACHROMAT (EXAMPLE 1)

f=100

| glass name | λ, pm | μ | γ | G | Θ | constrain type |
|---|---|---|---|---|---|---|
| FK 5 | 0.600000 | 1.487054 | 1.507505" | 24.816301 | 0.050234 | 1 |
| F6 | 0.600000 | 1.635261 | 1.686472 | 13,404796 | 0.1s5191 | 1 |
| FK 5 | 0.500000 | 1.491449 | 1.519843 | 18.307880 | 0.109634 | 2 |
| F6 | 0.500000 | 1.646817 | 1.724774 | 9.297117 | 0.381761 | 2 |

| Clement | φ (curvature) | 1/$ (radius) | glass | glass code |
|---|---|---|---|---|
| 1 | 0.390845E-01 | 0.255856E+02 | FK5 | 487704 |
| 2 | -0.142359E-01 | -0.702451E+02 | F6 | 636353 |

‘1 ABLE 3

OPTICAL CONSTANTS AND ('OBSTRUCTION PARAMETERS FOR A

THREE-ELEMENT THIN-LENS ACHROMAT (EXAMPLE 2)

f = 100”

| glass name | λ, μm | μ | γ | G | Θ |
|---|---|---|---|---|---|
| FK5 | 0,500000 | 1.491449 | 1.519843 | 18.307880 | 0.109634 |
| KzFS1 | 0.sooooo | 1,621067 | 1,679380 | 11.649486 | 0,257418 |
| SF15 | 0.500000 | 1.712489 | 1.814297 | 7.998414 | 0.s25363 |

| element | φ (curvature) | 1/+ (radius) | glass | glass code |
|---|---|---|---|---|
| 1 | 0.890669E-01 | 0.11 2275E+02 | FK5 | 487704 |
| 2 | -0.754849E-01 | -0.132477E+02 | KzFS1 | 613443 |
| 3 | ().183994E-01 | ().543498 E+02 | SF15 | 699.301 |

TABLE 4

CONSTRUCTION PARAMETERS FOR AN

EIGHT-ELEMENT THIN-LENS ACHROMAT (EXAMPLE 3)

f 1000

| element | $\phi$ | $1/\phi$ | glass | glass code |
|---|---|---|---|---|
| 1 | 0.2444361E-02 | 0.4091051E+03 | FK54 | 437907 |
| 2 | 0.843603E-02 | 0.118539E+03 | PK2 | 518651 |
| 3 | 0.492624E-01 | 0.202995E+02 | KS | 522595 |
| 4 | -0.492145E-01 | -0.203194E+02 | KF3 | 515547 |
| 5 | -0.507598E-01 | -0.197006E+02 | LLF2 | 541472 |
| 6 | 0.5002901E-01 | 0.199884E+02 | LF5 | 581409 |
| 7 | -0.105799E-01 | -0.945191E+02 | F6 | 636353 |
| 8 | 0.24620926E-03 | 0.40615857E+04 | SF59 | 952204 |

## CAPTIONS TO FIGURES

Fig. 1, Plot of $(\gamma - \mu)$ versus $(\mu - 1)$ for 203 Schott optical glasses and $CaF_2$ for wavelength 500 nm. The vectors $u_1$ and $u_2$ connecting the origin and the coordinates of F6 and FK5, respectively, define a triangle whose area is equal to one-half the value of det[A] for a lens constrained by Eqs. 7 and 9 and considered in Example 1 (see text).

Fig. 2. (Example 1 ) Wavelength dependence of focal length in (bin-lens approximation for a 2-element achromat (Table 2) in which the focal length is 100 at 600 nm and constrained to be a turning point (independent of wavelength) at 500 nm by Eq. 9: thus the phase power and the group power have the same value at 500 nm. In this and other illustrations both the phase focal length (heavy curve, from Eq. 7) and the group focal length (lighter curve, from Eq. 8) are shown

Fig. 3. (Example 2) Wavelength dependence of focal length in thin-lens approximation for a 3-element lens (Table 3). The first and second derivatives of power are constrained by Eqs. 9 and 17 to be zero at 500 nm, so that an inflexion (where the lens is achromatic and confocal in both phase and group power) is located al that wavelength Inset shows the lens as a cemented triplet at F/7.8 corrected for spherical aberration.

Fig. 4, (a) The stationary region of the group-achromatic triplet in Example 1 (Fig. 3). compared with (b) a standard triplet made of the same glasses. but constrained only by Eq. 7 to be confocal at three discrete wavelengths 480, 500 and 520 nm.

MORE

Fig. 5. (Example 3) Wavelength dependence of focal length in [bin-lc]is approximation for an 8-element lens (Table 4) in which the focal lengths are constrained by Eq. 7 to be 1000 at 500. 600, 700, and 800 nm, and also constrained by Eq. 9 to have turning points at the same four wavelengths.

END

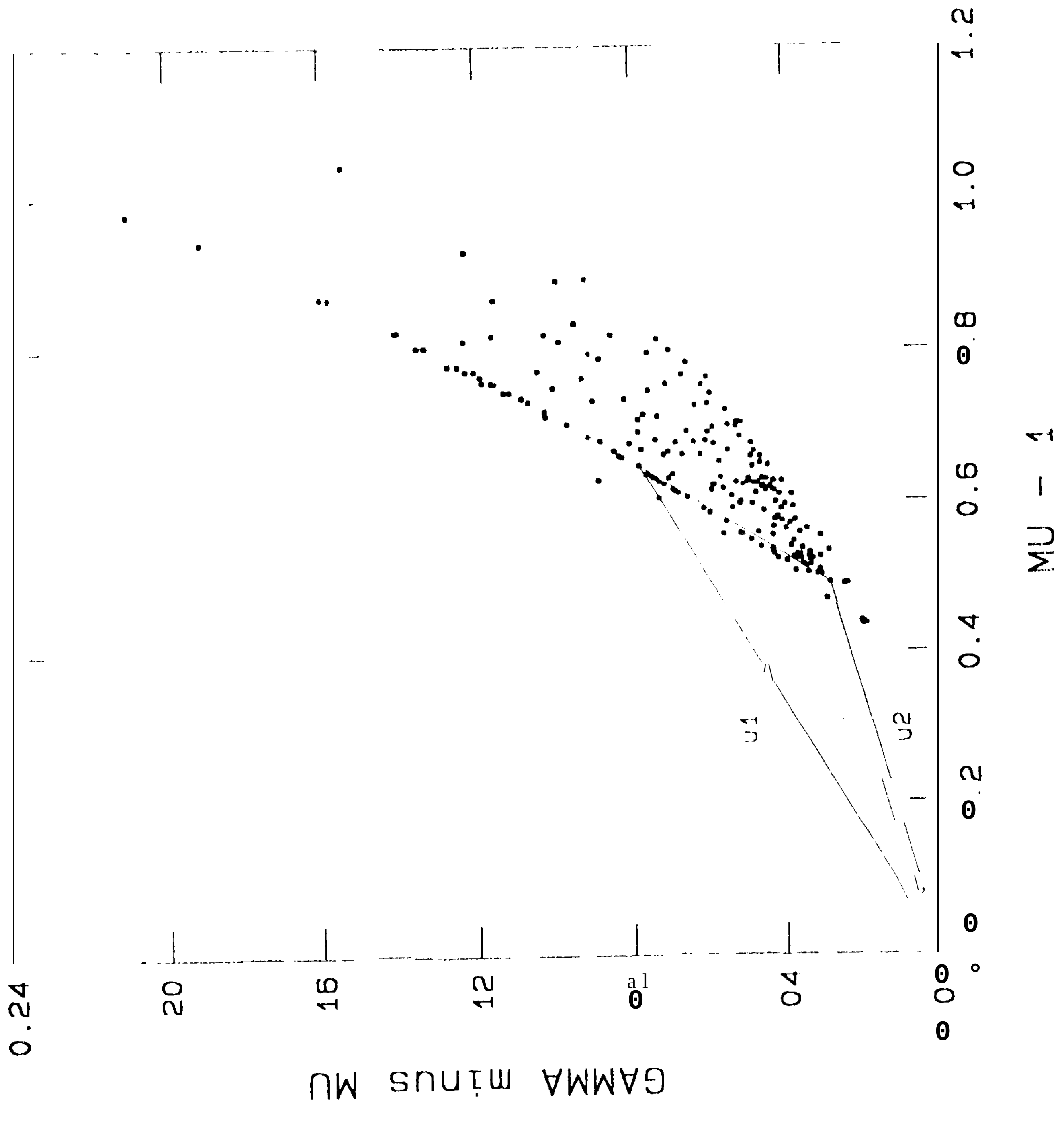
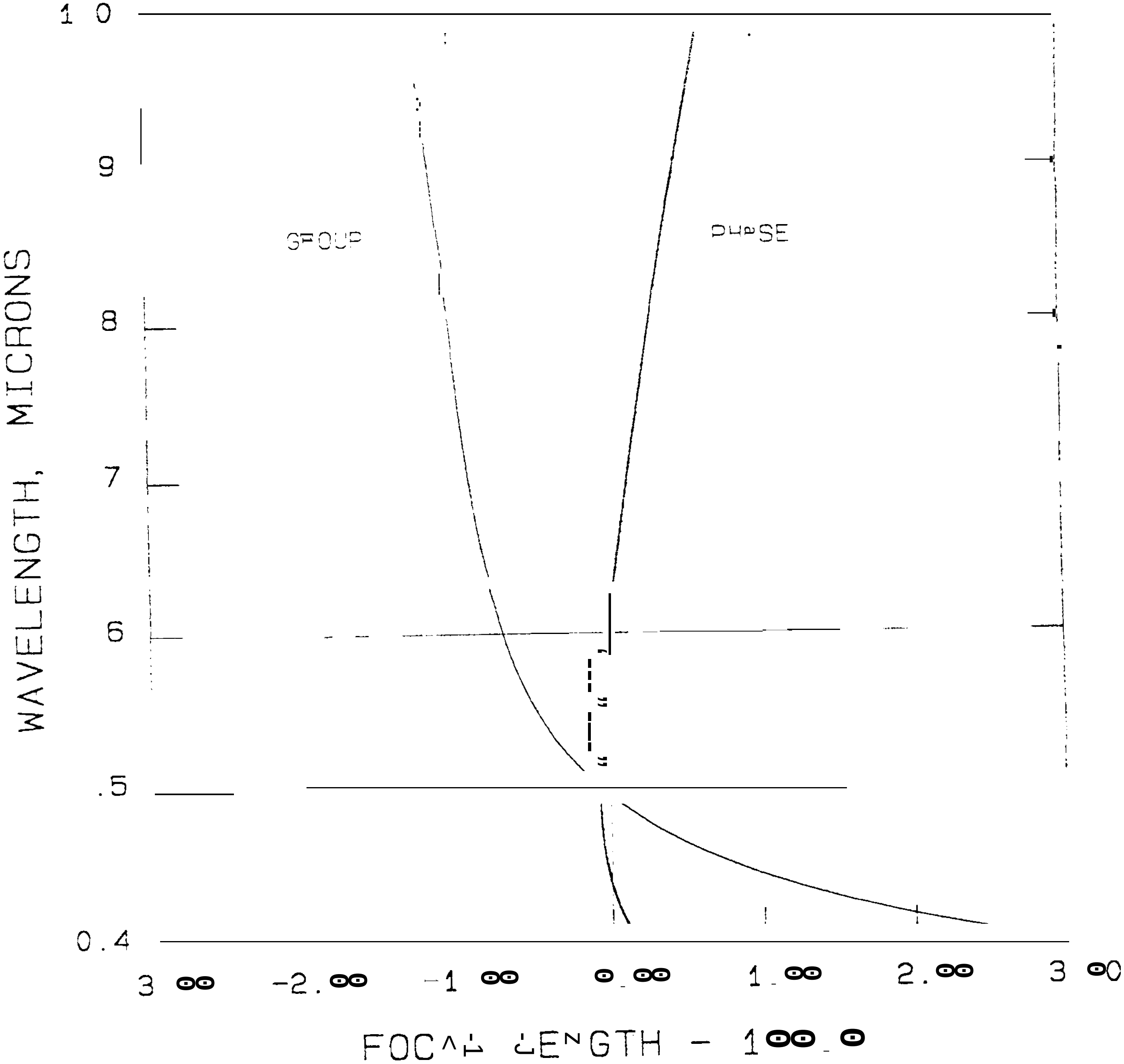



FIG 1



1.0
.9
.8
.7
.6
.5
0.4
WAVELENGTH, MICRONS
GROUP
PHASE
-3.00
-2.00
-1.00
0.00
1.00
2.00
3.00
FOCAL LENGTH - 100.0


FIG 2.

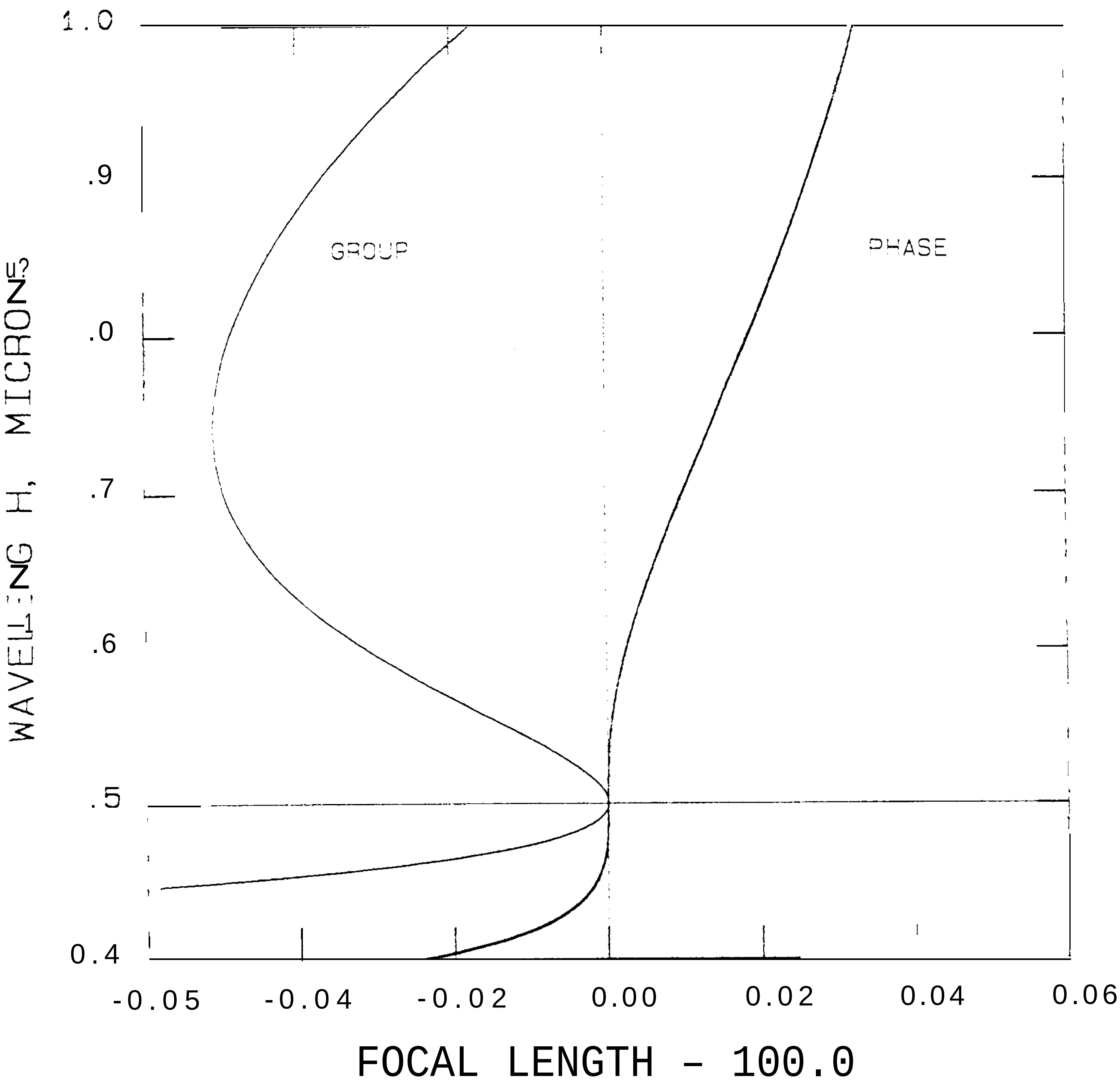

1.0
.9
.0
.7
.6
.5
0.4
WAVELENGTH, MICRONS
GROUP
PHASE
-0.05
-0.04
-0.02
0.00
0.02
0.04
0.06
FOCAL LENGTH – 100.0

Fig 3

INSET TO FIG 3

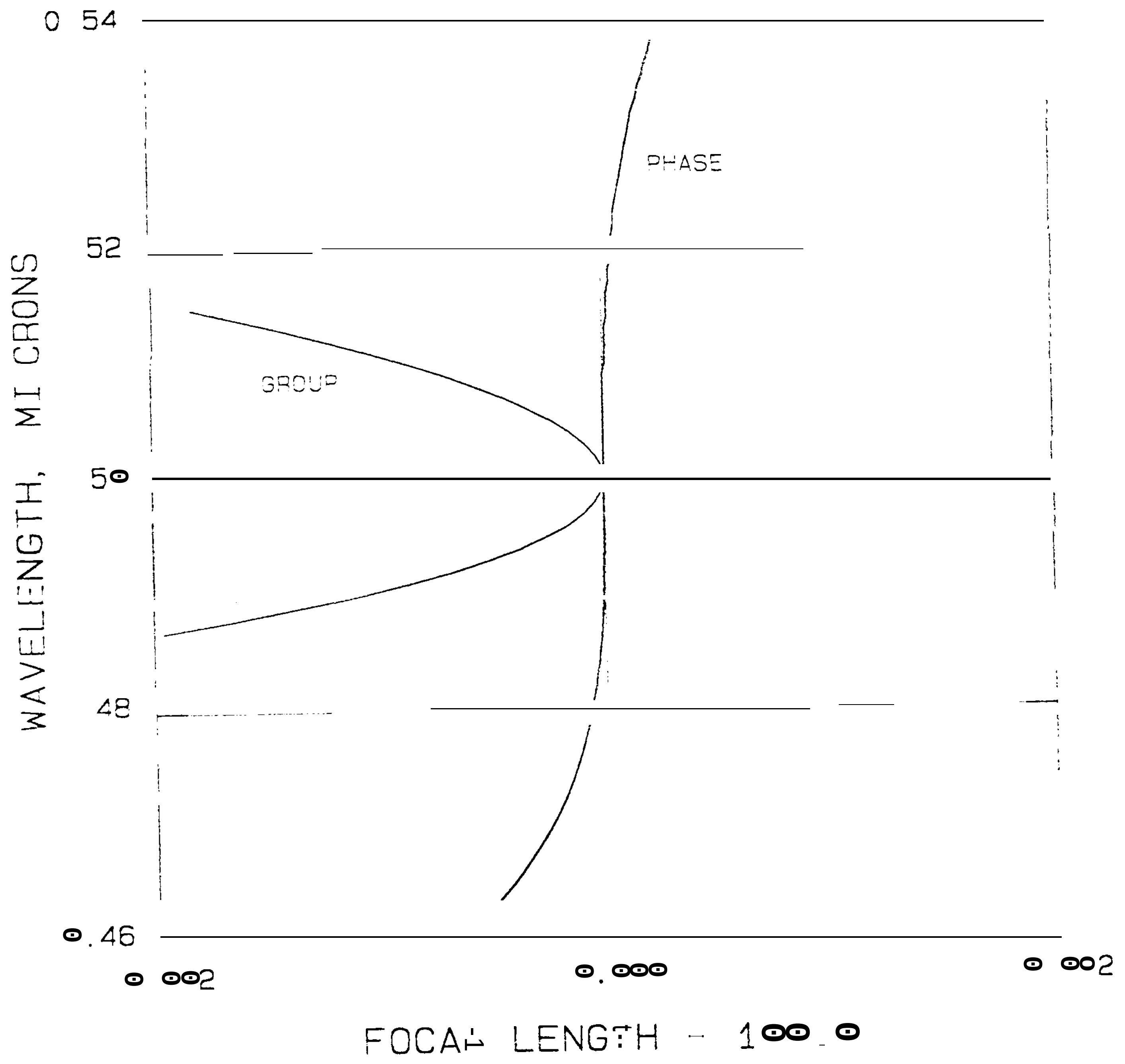

0 54
52
50
48
0.46
WAVELENGTH, MI CRONS
PHASE
GROUP
0 002
0.000
0 002
FOCAL LENGTH - 100.0

Fig 4a

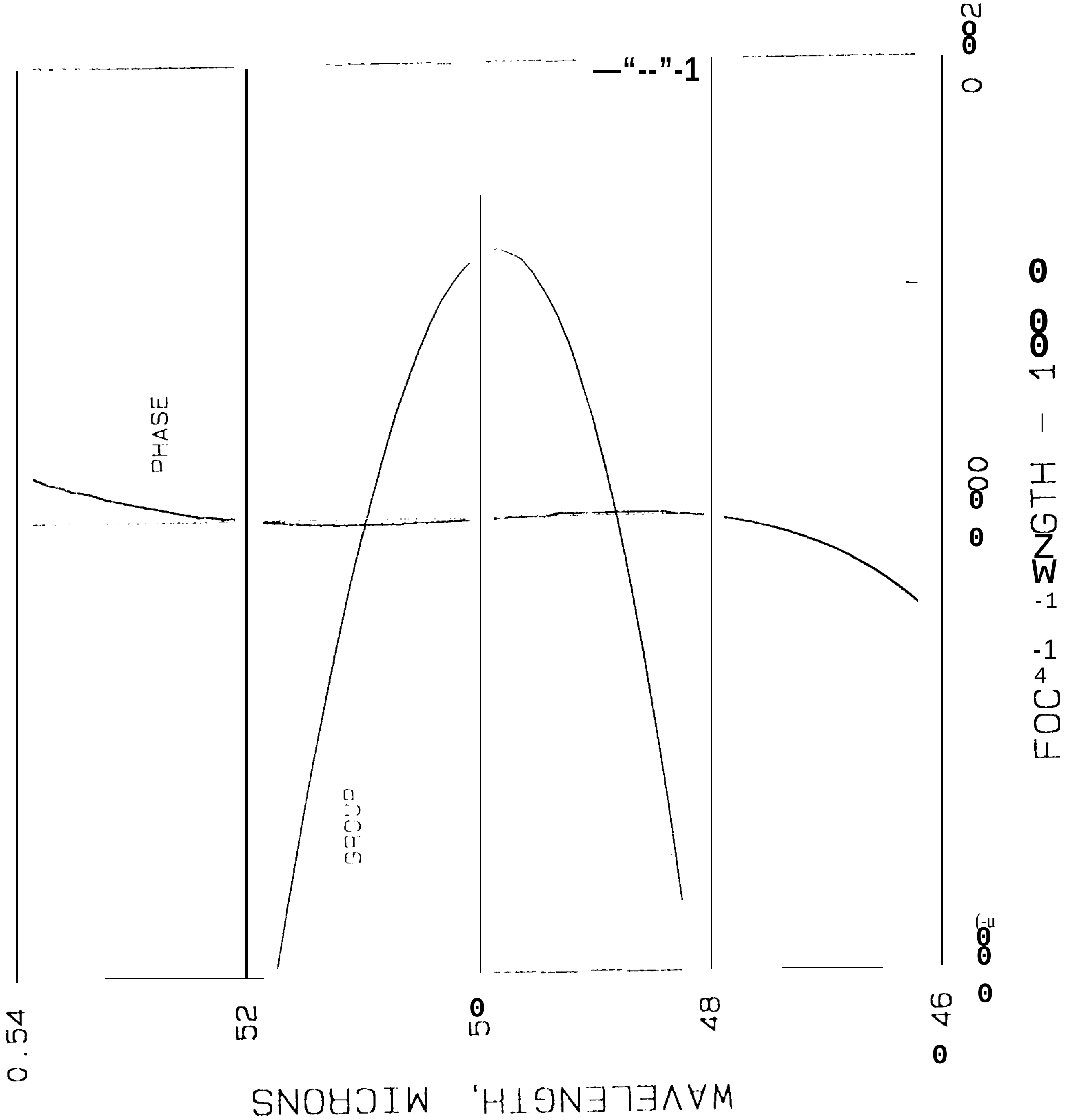



Fig 4b

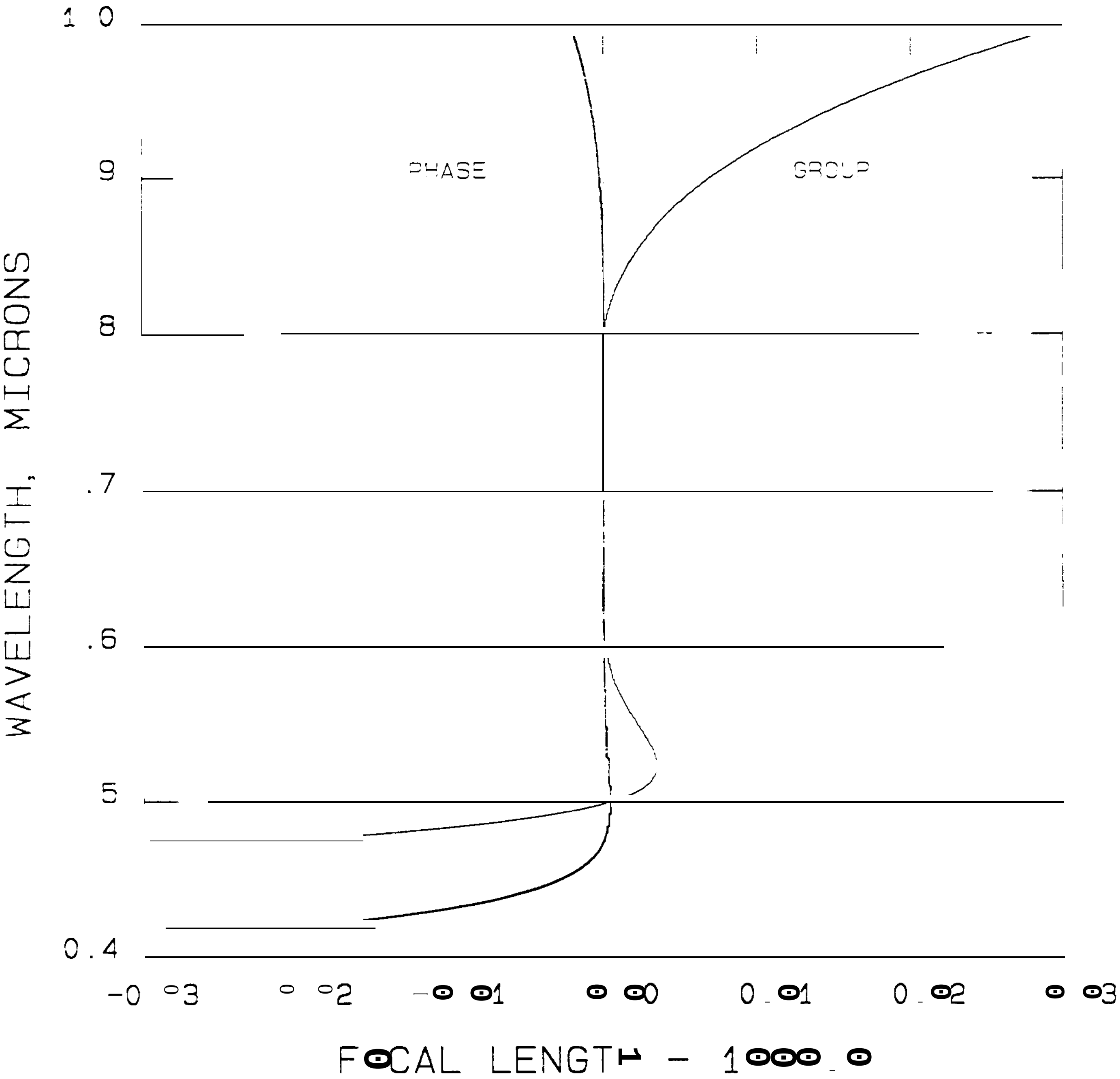

PHASE
GROUP
1.0
.9
.8
.7
.6
.5
0.4
WAVELENGTH, MICRONS
-0.03
-0.02
-0.01
0.00
0.01
0.02
0.03
FOCAL LENGTH - 1000.0

FIG 5